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The stability of d'Alembert and Jensen type functional equations ☆

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Abstract

The aim of this paper is to study the stability problem of the d'Alembert type and Jensen type functional equations:

$$f(x+y)+f(x+\sigma y)=2g(x)f(y),$$

$$f(x+y)+f(x+\sigma y)=2f(x)g(y),$$

$$f(x+y)+f(x+\sigma y)=2f(x).$$

Keywords: d'Alembert functional equation; Jensen functional equation; Cosine function; Superstability; Hyers–Ulam stability

1. Introduction

The Hyers–Ulam stability of the cosine functional equation

$$f(x+y)+f(x-y)=2f(x)f(y), \tag{A}$$

known as the *d'Alembert functional equation* was investigated by J.A. Baker in [3]. In [4], J. Baker, J. Lawrence and F. Zorzitto introduced the superstability of the exponential equation $f(x+y)=f(x)f(y)$. In light of this result, the stability of its functional equations has been

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investigated by Badora, Ger, Kannappan, Kim, Rassias, Sinopoulos, Stetkaer, etc. [1,5,7–10].

Badora and Ger [2] have improved the superstability of the d'Alembert equation (A) under the condition $|f(x+y)+f(x-y)-2f(x)f(y)| \leqslant \varphi(x)$ or $\varphi(y)$.

The aim of this paper is to investigate the superstability problem of the generalized d'Alembert type $(\tilde{A})$ and Jensen type $(\tilde{J})$ functional equations as follows:

$$f(x+y)+f(x+\sigma y)=2f(x)f(y), \tag{$\tilde{A}$}$$

$$f(x+y)+f(x+\sigma y)=2g(x)f(y), \tag{$\tilde{A}_{gf}$}$$

$$f(x+y)+f(x+\sigma y)=2f(x)g(y), \tag{$\tilde{A}_{fg}$}$$

$$f(x+y)+f(x+\sigma y)=2f(x). \tag{$\tilde{J}$}$$

In the special case $\sigma(x)=-x$ in the aforementioned functional equations, we will use notations A_{gf}, A_{fg}, that is, we will omit the tilde "˜". Also, A_g: $g(x+y)+g(x-y)=2g(x)g(y)$, the difference equation $\tilde{A}_{gf}(x,y)$: $f(x+y)+f(x+\sigma y)-2g(x)f(y)$.

In this paper, let $(G,+)$ be an Abelian group, $\mathbb{C}$ a complex number, $\mathbb{R}$ a real number, and let σ be an endomorphism of G with $\sigma(\sigma(x))=x$ for all $x\in G$. We will use $\sigma(x)=\sigma x$, and a homomorphism $m:G\to\mathbb{C}$ means $m(x+y)=m(x)+m(y)$. We may assume that f and g are nonzero functions and ε is a nonnegative real constant, $\varphi:G\to\mathbb{R}$. If all the results of this article are given by the Kannappan condition $f(x+y+z)=f(x+z+y)$, we will obtain identical results for the semigroup $(G,+)$.

2. Superstability of the generalized d'Alembert functional equations

In this section, we will investigate the stability of the generalized functional equations $(\tilde{A}_{gf})$ and $(\tilde{A}_{fg})$ for the d'Alembert functional equation (A).

Theorem 1. *Suppose that $f,g:G\to\mathbb{C}$ satisfy the inequality*

$$\bigl|f(x+y)+f(x+\sigma y)-2g(x)f(y)\bigr| \leqslant \begin{cases} \text{(i)}\ \varphi(x), \\ \text{(ii)}\ \varphi(y)\ \textit{and}\ \varphi(x), \end{cases} \tag{1}$$

for all $x,y\in G$. Then:

(i) *either f is bounded or g satisfies $(\tilde{A})$,*
(ii) *either g (or f) is bounded or g satisfies $(\tilde{A})$,*

also f and g satisfy $(\tilde{A}_{gf})$ and $(\tilde{A}_{fg})$. Further, in the latter case there exists a homomorphism m such that

$$f(x)=\frac{b}{2}\bigl(m(x)+m(\sigma x)\bigr) \quad \textit{and} \quad g(x)=\frac{1}{2}\bigl(m(x)+m(\sigma x)\bigr) \tag{2}$$

for all $x\in G$, where b is a constant.

Proof. For the case (i), let f be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that

$$0\neq\bigl|f(y_n)\bigr|\to\infty \quad \text{as } n\to\infty. \tag{3}$$

We will show that g satisfies $(\tilde{A})$. Taking $y=y_n$ in (1) we obtain

$$\left|\frac{f(x+y_n)+f(x+\sigma y_n)}{2f(y_n)}-g(x)\right| \leqslant \frac{\varphi(x)}{2|f(y_n)|},$$

that is,

$$\lim_{n\to\infty}\frac{f(x+y_n)+f(x+\sigma y_n)}{2f(y_n)}=g(x) \tag{4}$$

for all $x\in G$. Using (i) of (1) we have

$$\begin{aligned}&\bigl|f\bigl(x+(y+y_n)\bigr)+f\bigl(x+\sigma(y+y_n)\bigr)-2g(x)f(y+y_n)\\&\quad+f\bigl(x+(y+\sigma y_n)\bigr)+f\bigl(x+\sigma(y+\sigma y_n)\bigr)-2g(x)f(y+\sigma y_n)\bigr|\\&\leqslant 2\varphi(x)\end{aligned}$$

so that

$$\begin{aligned}&\biggl|\frac{f((x+y)+y_n)+f((x+y)+\sigma y_n)}{2f(y_n)}\\&\quad+\frac{f((x+\sigma y)+y_n)+f((x+\sigma y)+\sigma y_n)}{2f(y_n)}-2g(x)\frac{f(y+y_n)+f(y+\sigma y_n)}{2f(y_n)}\biggr|\\&\leqslant\frac{\varphi(x)}{|f(y_n)|}\end{aligned}$$

for all $x,y\in G$. By virtue of (4), we have

$$\bigl|g(x+y)+g(x+\sigma y)-2g(x)g(y)\bigr|\leqslant 0$$

for all $x,y\in G$. Therefore g satisfies $(\tilde{A})$.

For the proof of the case (ii), first we show that f (or g) is unbounded iff g (or f) is also unbounded. Putting $y=0$ in (ii) of (1) we obtain

$$\bigl|f(x)-g(x)f(0)\bigr|\leqslant\frac{\varphi(0)}{2} \tag{5}$$

for all $x\in G$. If g is bounded, then by (5), we have

$$\bigl|f(x)\bigr|=\bigl|f(x)-g(x)f(0)+g(x)f(0)\bigr|\leqslant\frac{\varphi(0)}{2}+\bigl|g(x)f(0)\bigr|,$$

which shows that f is also bounded. On the other hand, if f is bounded, we choose $y_0\in G$ such that $f(y_0)\neq 0$, and then by (1) we obtain

$$\bigl|g(x)\bigr|-\left|\frac{f(x+y_0)+f(x+\sigma y_0)}{2f(y_0)}\right|\leqslant\left|\frac{f(x+y_0)+f(x+\sigma y_0)}{2f(y_0)}-g(x)\right|\leqslant\frac{\varphi(y_0)}{2|f(y_0)|},$$

and it follows that g is also bounded on G.

Namely, if f (or g) is unbounded, then so is g (or f).

Let g be unbounded, then f is also unbounded. Then we can choose sequences $\{x_n\}$ and $\{y_n\}$ in G such that $g(x_n)\neq 0$ and $|g(x_n)|\to\infty$, $f(y_n)\neq 0$ and $|f(y_n)|\to\infty$ as $n\to\infty$.

Taking $x=x_n$ in (ii) of (1) we deduce

$$\lim_{n\to\infty}\frac{f(x_n+y)+f(x_n+\sigma y)}{2g(x_n)}=f(y) \tag{6}$$

for all $y\in G$. Using (1) we have

$$\begin{aligned}&\bigl|f\bigl((x_n+x)+y\bigr)+f\bigl((x_n+x)+\sigma y\bigr)-2g(x_n+x)f(y)\\&\quad+f\bigl((x_n+\sigma x)+y\bigr)+f\bigl((x_n+\sigma x)+\sigma y\bigr)-2g(x_n+\sigma x)f(y)\bigr|\\&\leqslant 2\varphi(y)\end{aligned}$$

for all $x, y \in G$ and every $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}&\left|\frac{f(x_n+(x+y))+f(x_n+\sigma(x+y))}{2g(x_n)}\right.\\ &\qquad\left.+\frac{f(x_n+(x+\sigma y))+f(x_n+\sigma(x+\sigma y))}{2g(x_n)}-2\cdot\frac{g(x_n+x)+g(x_n+\sigma x)}{2g(x_n)}f(y)\right|\\ &\quad\leqslant\frac{\varphi(y)}{|g(x_n)|}\end{aligned}$$

for all $x, y \in G$ and every $n \in \mathbb{N}$. Passing here to the limit as $n \to \infty$ with the use of $|g(x_n)| \to \infty$ and (6). Since g satisfies $(\tilde{A})$ by (i), f and g are solutions of $(\tilde{A}_{gf})$.

Applying (ii) of (1) again, we get

$$\begin{aligned}&\big|f\big((x_n+y)+x\big)+f\big((x_n+y)+\sigma x\big)-2g(x_n+y)f(x)\\ &\qquad+f\big((x_n+\sigma y)+x\big)+f\big((x_n+\sigma y)+\sigma x\big)-2g(x_n+\sigma y)f(x)\big|\\ &\quad\leqslant 2\varphi(x)\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{f(x_n+(x+y))+f(x_n+\sigma(x+y))}{2g(x_n)}\right.\\ &\qquad\left.+\frac{f(x_n+(x+\sigma y))+f(x_n+\sigma(x+\sigma y))}{2g(x_n)}-2f(x)\cdot\frac{g(x_n+y)+g(x_n+\sigma y)}{2g(x_n)}\right|\\ &\quad\leqslant\frac{\varphi(x)}{|g(x_n)|}\end{aligned}$$

for all $x, y \in G$ and every $n \in \mathbb{N}$.

Using (6) and the fact that g satisfies $(\tilde{A})$ by (i), the last inequality yields that f and g are solutions of $(\tilde{A}_{fg})$.

Finally, choose $y_0 \in G$ such that $f(y_0) \neq 0$. Then $(\tilde{A}_{gf})$ gives

$$g(x)=\frac{f(x+y_0)+f(x+\sigma y_0)}{2f(y_0)}.$$

Since g satisfies $(\tilde{A})$, from [6] we see that there exists a homomorphism $m: G \to \mathbb{C}$ satisfying the second part of (2). Using $(\tilde{A}_{gf})$ and $(\tilde{A}_{fg})$, it is easy to see that $f(x) = bg(x)$, for some constant b. Therefore the proof of the theorem is complete. □

In the case of $\sigma(x) = -x$ in Theorem 1, we can obtain the following corollary.

Corollary 2. *Suppose that $f, g: G \to \mathbb{C}$ satisfy the inequality*

$$\big|f(x+y)+f(x-y)-2g(x)f(y)\big|\leqslant\begin{cases}\text{(i)} & \varphi(x),\\ \text{(ii)} & \varphi(y)\ \textit{and}\ \varphi(x),\end{cases}$$

for all $x, y \in G$.

Then:

(i) *either f is bounded or g satisfies (A),*
(ii) *either g (or f) is bounded or g satisfies (A),*

also f and g satisfy (A_{gf}) and (A_{fg}). Further, in the latter case there exists a homomorphism m such that

$$f(x) = \frac{b}{2}\big(m(x) + m(-x)\big) \quad \textit{and} \quad g(x) = \frac{1}{2}\big(m(x) + m(-x)\big)$$

for all $x \in G$, where b is a constant.

If $f = g$ in Theorem 1, then the stability problem of the functional equation $(\tilde{A})$ is proved as a corollary.

Corollary 3. *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$\big|f(x+y) + f(x+\sigma y) - 2f(x)f(y)\big| \leqslant \begin{cases} \text{(i)} \ \varphi(x), \\ \text{(ii)} \ \varphi(y) \textit{ and } \varphi(x), \end{cases}$$

for all $x, y \in G$. Then, in all cases (i) *and* (ii)*, either f is bounded or f satisfies $(\tilde{A})$. Further, there exists a homomorphism m such that*

$$f(x) = \frac{b}{2}\big(m(x) + m(\sigma x)\big)$$

for all $x \in G$, where b is a constant.

Remark 1. The results of Theorem 1 imply the following six known theorems as corollaries, and also the combinations by cases of conditions ($g = f$, $g \neq f$, $\sigma(x) = -x$, $\sigma(x) = x$, $\varphi(x) \neq \varphi(y) \neq c$, $\varphi(x) = \varphi(y) = c$) imply us other corollaries.

(i) If $g = f$ and $\sigma(x) = -x$, then Eqs. $(\tilde{A}_{gf})$ and $(\tilde{A}_{fg})$ imply (A). The stability of (A) in cases (i) and (ii) of Corollary 3 was proved by Badora and Ger in [2].
(ii) The results in the case $\varphi(x) = c$: constant in Theorem 1 were found by Kim and Lee [8].
(iii) The results in the case of $\varphi(x) = c$ and $\sigma(x) = -x$ in Theorem 1 were discovered by Kannappan and Kim [7].
(iv) If $\sigma(x) = -x$, then Eq. $(\tilde{A})$ implies (A). The stability of (A) when bounded by a constant has been proved by Badora [1] and Baker [3].
(v) If $\sigma(x) = x$, then Eq. $(\tilde{A})$ implies the exponential functional equation $f(x+y) = f(x)f(y)$. The stability of this equation was demonstrated in Baker, Lawrence and Zorzitto [4], and many other papers.
(vi) If $\sigma(x) = x^{-1}$ and the operation of a group G is multiplication, then Eq. $(\tilde{A})$ implies $f(xy) + f(xy^{-1}) = 2f(x)f(y)$. This equation was discovered by Kannappan [6].

We will be prove the stability of $(\tilde{A}_{fg})$ using a strategy similar to that of Theorem 1.

Theorem 4. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\big|f(x+y) + f(x+\sigma y) - 2f(x)g(y)\big| \leqslant \begin{cases} \text{(i)} \ \varphi(y), \\ \text{(ii)} \ \varphi(x) \textit{ and } \varphi(y), \end{cases} \tag{7}$$

for all $x, y \in G$. Then

(i) *either f is bounded or g satisfies $(\tilde{A})$,*
(ii) *either g (or f) with $f(\sigma x) = f(x)$ is bounded or g satisfies $(\tilde{A})$,*

also f and g satisfy $(\tilde{A}_{fg})$ and $(\tilde{A}_{gf})$. Further, in the latter case there exists a homomorphism m which satisfies (2).

Proof. For the case (i), let f be unbounded. Then we can choose a sequence $\{x_n\}$ in G such that

$$0 \neq \left|f(x_n)\right| \to \infty, \quad \text{as } n \to \infty. \tag{8}$$

We will show that g satisfies $(\tilde{A})$. Taking $x = x_n$ in (7) we obtain

$$\lim_{n\to\infty} \frac{f(x_n + y) + f(x_n + \sigma y)}{2f(x_n)} = g(y) \quad \forall y \in G. \tag{9}$$

Using (i) of (7) we have

$$\begin{aligned}&\big|f\big((x_n + x) + y\big) + f\big((x_n + x) + \sigma y\big) - 2f(x_n + x)g(y)\\ &\qquad + f\big((x_n + \sigma x) + y\big) + f\big((x_n + \sigma x) + \sigma y\big) - 2f(x_n + \sigma x)g(y)\big|\\ &\quad \leqslant 2\varphi(y)\end{aligned}$$

so that

$$\begin{aligned}&\bigg|\frac{f(x_n + (x + y)) + f(x_n + \sigma(x + y))}{2f(x_n)}\\ &\qquad + \frac{f(x_n + (x + \sigma y)) + f(x_n + \sigma(x + \sigma y))}{2f(x_n)} - 2\frac{f(x_n + x) + f(x_n + \sigma x)}{2f(x_n)}g(y)\bigg|\\ &\quad \leqslant \frac{\varphi(y)}{|f(x_n)|}\end{aligned}$$

for all $x, y \in G$. By virtue of (8) and (9), we have

$$\big|g(x + y) + g(x + \sigma y) - 2g(x)g(y)\big| \leqslant 0$$

for all $x, y \in G$. Therefore g satisfies $(\tilde{A})$.

For the case (ii), we can see that, similar to Theorem 1, f (or g) is unbounded iff g (or f) is also unbounded. Namely, if f is bounded, choose $x_0 \in G$ such that $f(x_0) \neq 0$ and use (ii) of (7) to get

$$\big|g(y)\big| - \frac{|f(x_0 + y) + f(x_0 + \sigma y)|}{2|f(x_0)|} \leqslant \left|\frac{f(x_0 + y) + f(x_0 + \sigma y)}{2f(x_0)} - g(y)\right| \leqslant \frac{\varphi(x_0)}{2|f(x_0)|},$$

which shows that g is also bounded.

Suppose f is unbounded. Putting $x = 0$ in (ii) of (7), we have $|f(y) + f(\sigma y) - 2f(0)g(y)| \leqslant \varphi(0)$, that is, $|f(y) - f(0)g(y)| \leqslant \frac{\varphi(0)}{2}$ since $f(\sigma x) = f(x)$ for all $x \in G$. This implies that g is also unbounded.

Let g be unbounded, then f is also unbounded. Then we can choose sequences $\{x_n\}$ and $\{y_n\}$ in G such that $f(x_n) \neq 0$ and $|f(x_n)| \to \infty$, $g(y_n) \neq 0$ and $|g(y_n)| \to \infty$ as $n \to \infty$.

Taking $y = y_n$ in (ii) of (7) we deduce

$$\lim_{n\to\infty} \frac{f(x + y_n) + f(x + \sigma y_n)}{2g(y_n)} = f(x) \tag{10}$$

for all $x \in G$. Again applying (ii) of (7) we have

$$\begin{aligned}&\bigl|f\bigl(x+(y+y_n)\bigr)+f\bigl(x+\sigma(y+y_n)\bigr)-2f(x)g(y+y_n)\\&\qquad+f\bigl(x+(y+\sigma y_n)\bigr)+f\bigl(x+\sigma(y+\sigma y_n)\bigr)-2f(x)g(y+\sigma y_n)\bigr|\\&\quad\leqslant 2\varphi(x)\end{aligned}$$

so that

$$\begin{aligned}&\Biggl|\frac{f((x+y)+y_n)+f((x+y)+\sigma y_n)}{2g(y_n)}\\&\qquad+\frac{f((x+\sigma y)+y_n)+f((x+\sigma y)+\sigma y_n)}{2g(y_n)}-2f(x)\frac{g(y_n+y)+g(y_n+\sigma y)}{2g(y_n)}\Biggr|\\&\quad\leqslant\frac{\varphi(x)}{|g(y_n)|}\end{aligned}$$

for all $x, y \in G$. Since g satisfies $(\tilde{A})$, it follows from (10) that

$$\bigl|f(x+y)+f(x+\sigma y)-2f(x)g(y)\bigr|\leqslant 0$$

for all $x, y \in G$. Hence f and g are solutions of $(\tilde{A}_{fg})$.

Using (ii) of (7) we have

$$\begin{aligned}&\bigl|f\bigl(y+(x+y_n)\bigr)+f\bigl(y+\sigma(x+y_n)\bigr)-2f(y)g(x+y_n)\\&\qquad+f\bigl(y+(x+\sigma y_n)\bigr)+f\bigl(y+\sigma(x+\sigma y_n)\bigr)-2f(y)g(x+\sigma y_n)\bigr|\\&\quad\leqslant 2\varphi(y)\end{aligned}$$

for all $x, y \in G$. Since $f(\sigma x) = f(x)$ for all $x \in G$, we have

$$\begin{aligned}&\Biggl|\frac{f((x+y)+y_n)+f((x+y)+\sigma y_n)}{2g(y_n)}\\&\qquad+\frac{f((x+\sigma y)+y_n)+f((x+\sigma y)+\sigma y_n)}{2g(y_n)}-2f(y)\frac{g(y_n+x)+g(y_n+\sigma x)}{2g(y_n)}\Biggr|\\&\quad\leqslant\frac{\varphi(y)}{|g(y_n)|}\end{aligned}$$

for all $x, y \in G$. Since g satisfies $(\tilde{A})$, using (10), we have

$$\bigl|f(x+y)+f(x+\sigma y)-2g(x)f(y)\bigr|\leqslant 0$$

for all $x, y \in G$. Therefore f and g are solutions of $(\tilde{A}_{gf})$.

Since g satisfies $(\tilde{A})$, the existence of a homomorphism m follows from the result of [6]. Therefore the proof of the theorem is complete. □

Remark 2. In particular, if we apply the combination of cases

(a) $g = f$, or $g \neq f$,
(b) $\sigma(x)$, $\sigma(x) = -x$, or $\sigma(x) = x$,
(c) $\varphi(x) = \varphi(y) = c$, or $\varphi(x) \neq \varphi(y) \neq c$

to Theorem 4, we obtain the results of the papers [1–8] as in Remark 1.

Remark 3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions with $f(x) = x$ and $g(x) \equiv 1$ for all $x \in \mathbb{R}$, and let $\sigma(x) = -x$. Then $|f(x+y) + f(x-y) - 2f(x)g(y)| = 0$, but f is unbounded and f, g do not satisfy Eq. $(\tilde{A}_{gf})$, that is, $|f(x+y) + f(x-y) - 2g(x)f(y)| \neq 0$. This shows that the condition $f(\sigma x) = f(x)$ is essential in case (ii) of Theorem 4.

In [1] Badora gave a counter-example to illustrate the failure of the superstability of the cosine functional equation $(\tilde{A})$ in the case of vector-valued mappings. The following example shows that Theorems 1 and 4 are not true for vector-valued mappings.

Example. Let f and g be unbounded solutions of $(\tilde{A}_{gf})$ (or $(\tilde{A}_{fg})$) where $f, g : G \to \mathbb{C}$. Define $f_1, g_1 : G \to M_2(\mathbb{C})$ (2×2 matrices over $\mathbb{C}$) by

$$f_1(x) = \begin{pmatrix} f(x) & 0 \\ 0 & \alpha_1 \end{pmatrix}, \qquad g_1(x) = \begin{pmatrix} g(x) & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

for all $x \in G$ where $\alpha_1 \neq 0$, $\alpha_2 \neq 1$. Then

$$\left\| f_1(x+y) + f_1(x+\sigma y) - 2f_1(y)g_1(x) \right\| = \text{constant} > 0$$

(or $\| f_1(x+y) + f_1(x+\sigma y) - 2f_1(x)g_1(y) \| = \text{constant} > 0$) for all $x, y \in G$. These f_1 and g_1 are not bounded and do not satisfy $(\tilde{A})$.

Each case of the following theorem will be proved by an application to either Theorem 1 or Theorem 4.

Theorem 5. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ and $\varphi : G \to \mathbb{R}$ satisfy one of the inequalities*

$$\left\| f(x+y) + f(x+\sigma y) - 2g(x)f(y) \right\| \leqslant \begin{cases} \text{(i)} \ \varphi(x), \\ \text{(ii)} \ \varphi(y) \text{ and } \varphi(x), \end{cases} \qquad \forall x, y \in G, \tag{11}$$

or

$$\left\| f(x+y) + f(x+\sigma y) - 2f(x)g(y) \right\| \leqslant \begin{cases} \text{(i)} \ \varphi(y), \\ \text{(ii)} \ \varphi(x) \text{ and } \varphi(y), \end{cases} \qquad \forall x, y \in G, \tag{12}$$

with $f(\sigma x) = f(x)$ in case (ii) *of* (12).

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(a) *if the superposition $x^* \circ f$ fails to be bounded, then $(\tilde{A})$ provides in each case* (i) *of* (11) *and* (12),
(b) *if the superposition $x^* \circ g$ fails to be bounded, then $(\tilde{A})$, $(\tilde{A}_{fg})$ and $(\tilde{A}_{gf})$ provide in each case* (ii) *of* (11) *and* (12).

Proof. The proofs of each case are very similar, so it suffices to show the proof of the case (ii) of (11) in (b). Assume that (ii) of (11) holds and fix arbitrarily a linear multiplicative functional $x^* \in E$. As is well known we have $\|x^*\| = 1$ whence, for every $x, y \in G$, we have

$$\begin{aligned} \varphi(y) &\geqslant \left\| f(x+y) + f(x+\sigma y) - 2g(x)f(y) \right\| \\ &= \sup_{\|y^*\|=1} \left| y^*\big(f(x+y) + f(x+\sigma y) - 2g(x)f(y)\big) \right| \\ &\geqslant \left| x^*\big(f(x+y)\big) + x^*\big(f(x+\sigma y)\big) - 2x^*\big(g(x)\big)x^*\big(f(y)\big) \right|, \end{aligned}$$

which states that the superposition $x^* \circ g$ and $x^* \circ f$ yields a solution of inequality (11). Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 1 shows that the functions $x^* \circ g$ and $x^* \circ f$ solve the generalized d'Alembert's equation $(\tilde{A}_{gf})$. In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the generalized d'Alembert's difference $\tilde{A}_{gf}(x, y)$ falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\tilde{A}_{gf}(x, y) \in \bigcap \{\ker x^*:\ x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.,

$$f(x+y) + f(x+\sigma y) - 2g(x)f(y) = 0 \quad \text{for all } x, y \in G,$$

as claimed. The other cases are similar. $\square$

3. Stability of the Jensen type functional equation

Let $g(x) \equiv k$ in $(\tilde{A}_{gf})$. Then we have $f(x+y) + f(x+\sigma y) = 2kf(y)$ for all $x, y \in G$. Putting $y = 0$ in this equation we have $f(x) = kf(0)$. Hence f is a constant function.

Let $g(y) \equiv 1$ in $(\tilde{A}_{fg})$. Then we have the Jensen type functional equation $(\tilde{J})$ for the *Jensen functional equation*

$$f(x+y) + f(x-y) = 2f(x) \tag{J}$$

for all $x, y \in G$. If $f(\sigma x) = f(x)$ for all $x \in G$, then Eq. $(\tilde{J})$ implies that $f(x) = f(0)$ for all $x \in G$. P. Sinopoulos [10] determined the general solution of the functional equation $(\tilde{J})$.

Now, we prove the stability of Jensen (J) and Jensen type $(\tilde{J})$ functional equations. We show that a general solution of the Jensen type $(\tilde{J})$ functional equation is represented by a summation of the additive mapping and a constant.

Theorem 6. *Let $(G, +)$ be a group and E a Banach space. Suppose that $f : G \to E$ satisfies the inequality*

$$\| f(x+y) + f(x+\sigma y) - 2f(x) \| \leqslant \varphi(x), \tag{13}$$

where φ satisfies $\Phi(x) := \sum_{k=1}^{\infty} \frac{1}{2^k}\varphi(2^{k-1}x) < \infty$. Then there exists a unique additive mapping $A : G \to E$ as a solution of $(\tilde{J})$ such that $A(\sigma x) = -A(x)$ and

$$\| f(x) - f(0) - A(x) \| \leqslant \Phi(x) + \frac{1}{2}\varphi(0) \tag{14}$$

for all $x \in E$.

Proof. Putting $y = x$ in (13) we have

$$\| f(2x) + f(x+\sigma x) - 2f(x) \| \leqslant \varphi(x) \tag{15}$$

for all $x \in G$. Putting $x = 0$ in (13) and replacing y by $x + \sigma x$ we have

$$\| f(x+\sigma x) - f(0) \| \leqslant \frac{1}{2}\varphi(0) \tag{16}$$

for all $x \in G$. By (15) and (16) we have

$$\| f(2x) - 2f(x) + f(0) \| \leqslant \varphi(x) + \frac{1}{2}\varphi(0)$$

for all $x \in G$. Let $F(x) := f(x) - f(0)$ for all $x \in G$. Then $F(0) = 0$ and

$$\|F(2x) - 2F(x)\| \leqslant \varphi(x) + \frac{1}{2}\varphi(0) \tag{17}$$

for all $x \in G$. Replacing x by $2^n x$ in (17) and dividing its result by 2^{n+1} we get

$$\left\|\frac{F(2^n x)}{2^n} - \frac{F(2^{n+1}x)}{2^{n+1}}\right\| \leqslant \frac{1}{2^{n+1}} \cdot \left\{\varphi\left(2^n x\right) + \frac{1}{2}\varphi(0)\right\} \tag{18}$$

for all $x \in E$ and all nonnegative integers n. Using (18) and the triangle inequality we have

$$\left\|\frac{F(2^m x)}{2^m} - \frac{F(2^n x)}{2^n}\right\| \leqslant \sum_{k=m+1}^{n} \frac{1}{2^k}\left\{\varphi\left(2^{k-1}x\right) + \frac{1}{2}\varphi(0)\right\} \tag{19}$$

for all $x \in E$ and all nonnegative integers m and n with $m < n$. This shows that $\{\frac{F(2^n x)}{2^n}\}$ is a Cauchy sequence for all $x \in E$ since the right side of (19) converges to zero by the assumption of φ when $m \to \infty$. Consequently, we can define a mapping $A : G \to E$ by

$$A(x) := \lim_{n\to\infty} \frac{F(2^n x)}{2^n}$$

for all $x \in G$. Putting $m = 0$ in (19) and taking the limit as $n \to \infty$, we obtain (14). Also, we get $A(0) = 0$ and

$$\begin{aligned}
&\|A(x+y) + A(x+\sigma y) - 2A(x)\| \\
&\quad = \lim_{n\to\infty}\left\|\frac{F(2^n x + 2^n y)}{2^n} + \frac{F(2^n x + 2^n \sigma y)}{2^n} - 2\frac{F(2^n x)}{2^n}\right\| \\
&\quad = \lim_{n\to\infty}\frac{1}{2^n}\left\|f\left(2^n x + 2^n y\right) + f\left(2^n x + 2^n \sigma y\right) - 2f\left(2^n x\right)\right\| \\
&\quad \leqslant \lim_{n\to\infty} 2 \cdot \frac{1}{2^{n+1}}\varphi\left(2^n x\right) = 0
\end{aligned}$$

for all $x, y \in G$, which satisfies Eq. $(\tilde{J})$. It also follows that A is additive with $A(\sigma x) = -A(x)$ (cf. [10]).

Now, let $A' : G \to E$ be another additive mapping satisfying (14). Then we have

$$\begin{aligned}
&\|A(x) - A'(x)\| \\
&\quad = 2^{-n}\left\|A\left(2^n x\right) - A'\left(2^n x\right)\right\| \\
&\quad \leqslant 2^{-n}\left(\left\|A\left(2^n x\right) - f\left(2^n x\right) + f(0)\right\| + \left\|A'\left(2^n x\right) - f\left(2^n x\right) + f(0)\right\|\right) \\
&\quad \leqslant 2 \cdot \sum_{k=n+1}^{\infty} \frac{1}{2^k}\left\{\varphi\left(2^{k-1}x\right) + \frac{1}{2}\varphi(0)\right\}
\end{aligned} \tag{20}$$

for all $x \in E$ and all positive integers n. Taking the limit in (20) as $n \to \infty$, we can conclude that $A(x) = A'(x)$ for all $x \in E$. This proves the uniqueness of A. □

Theorem 7. *Let $(G, +)$ be a group and E a Banach space. Suppose that $f : G \to E$ satisfies the inequality*

$$\|f(x+y) + f(x+\sigma y) - 2f(x)\| \leqslant \varphi(y), \tag{21}$$

where φ satisfies $\varphi(\sigma y) = -\varphi(y)$ and $\Phi(y) := \sum_{k=1}^{\infty} \frac{1}{2^k}\varphi(2^{k-1}y) < \infty$.

Then there exists a unique additive mapping $A : G \to E$ as a solution of $(\tilde{J})$ such that $A(\sigma x) = -A(x)$ and

$$\left\| f(x) - f(0) - A(x) \right\| \leqslant \Phi(x) \tag{22}$$

for all $x \in E$.

Proof. Putting $x = 0$ in (21) and replacing y by $x + \sigma x$ we have

$$\left\| f(x + \sigma x) - f(0) \right\| \leqslant \frac{1}{2}\varphi(x + \sigma x)$$

for all $x \in G$. Since the condition $\varphi(\sigma y) = -\varphi(y)$ implies $\varphi(x + \sigma x) = 0$, we have

$$f(x + \sigma x) = f(0). \tag{23}$$

Putting $y = x$ in (21), by (23) we have

$$\left\| f(2x) - 2f(x) + f(0) \right\| \leqslant \varphi(x) \tag{24}$$

for all $x \in G$. The remainder of the proof proceeds similarly to that in Theorem 6. □

From Theorems 6 and 7, we can obtain the following four corollaries with the cases $\sigma(x) = -x$ and $\varphi(x) = \varepsilon$ as natural results.

Corollary 8. *Let $(G, +)$ be a group and E a Banach space. Suppose that $f : G \to E$ satisfies the inequality*

$$\left\| f(x + y) + f(x - y) - 2f(x) \right\| \leqslant \varphi(x), \tag{25}$$

where φ satisfies $\Phi(x) := \sum_{k=1}^{\infty} \frac{1}{2^k}\varphi(2^{k-1}x) < \infty$.

Then there exists a unique additive mapping $A : G \to E$ as a solution of (J) such that $A(-x) = -A(x)$ and

$$\left\| f(x) - f(0) - A(x) \right\| \leqslant \Phi(x)$$

for all $x \in E$.

Proof. Putting $y = x$ in (25), we immediately have the inequality (24). The remainder of proof follows as in Theorem 6. □

Corollary 9. *Let $(G, +)$ be a group and E a Banach space. Suppose that $f : G \to E$ satisfies the inequality*

$$\left\| f(x + y) + f(x - y) - 2f(x) \right\| \leqslant \varphi(y),$$

where φ satisfies $\Phi(y) := \sum_{k=1}^{\infty} \frac{1}{2^k}\varphi(2^{k-1}y) < \infty$.

Then there exists a unique additive mapping $A : G \to E$ as a solution of (J) such that $A(-x) = -A(x)$ and

$$\left\| f(x) - f(0) - A(x) \right\| \leqslant \Phi(x)$$

for all $x \in E$.

Proof. The proof is the same as the one for Corollary 8. □

Corollary 10. (See [8, Theorem 5].) *Let $(G, +)$ be a group and E a Banach space. Suppose that $f : G \to E$ satisfies the inequality*

$$\left\| f(x+y) + f(x+\sigma y) - 2f(x) \right\| \leqslant \varepsilon,$$

then there exists a unique additive mapping $A : G \to E$ as a solution of (J) such that $A(\sigma x) = -A(x)$ and

$$\left\| f(x) - f(0) - A(x) \right\| \leqslant \frac{3}{2}\varepsilon$$

for all $x \in E$.

Proof. The proof follows on putting $\varphi(x) = \varepsilon$ in Theorem 6. □

Corollary 11. *Let $(G, +)$ be a group and E a Banach space. Suppose that $f : G \to E$ satisfies the inequality*

$$\left\| f(x+y) + f(x-y) - 2f(x) \right\| \leqslant \varepsilon, \tag{26}$$

then there exists a unique additive mapping $A : G \to E$ as a solution of (J) such that $A(-x) = -A(x)$ and

$$\left\| f(x) - f(0) - A(x) \right\| \leqslant \varepsilon$$

for all $x \in E$.

Proof. We put $y = 0$ in (26), and then, by Corollary 8, the result follows. □

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